# NAKED SINGULARITIES AND THE WILSON LOOP

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We give observations about dualities where one of the dual theories is geometric. These are illustrated with a duality between the simple harmonic oscillator and a topological field theory. We then discuss the Wilson loop in the context of the AdS/CFT duality. We show that the Wilson loop calculation for certain asymptotically AdS scalar field spacetimes with naked singularities gives results qualitatively similar to that for the AdS black hole. In particular, it is apparent that (dimensional) metric parameters in the singular spacetimes permit a "thermal screening" interpretation for the quark potential in the boundary theory, just like black hole mass. This suggests that the Wilson loop calculation merely captures metric parameter information rather than true horizon information.

*Keywords*: Dualities, AdS/CFT correspondence, Wilson loops, singular spacetimes.

## 1. Introduction

The basic idea underlying the term 'duality" may be stated as follows: Two distinct classical theories, prescribed by actions based on different sets of fields, not necessarily in the same spacetime dimension, may be equivalent at the quantum level. This means that it is possible to establish a correspondence between observables and states in the respective quantum theories. If the difference in spacetime dimensions of the dual theories is $\pm 1$, then the term "holography" is used to capture the general idea that all physical information of the higher dimensional theory is potentially manifested in one lower dimension.

That such correspondences may be possible should not come as a surprise since quantum theories know about spacetime dimension only through the choice of representation chosen for classical observables. In particular the coordinate and momentum representations are the specific choices which inject spacetime dimension information into quantum theory. Their use is not necessary since even the simple harmonic oscillator may be quantized algebraically via the creation and annihilation operators.

In theories with large gauge invariance groups, particularly general coordinate invariance, the interest lies in identifying and using gauge invariant observables for quantization. In such theories it is not appropriate to quantize based on the fundamental Poisson bracket, but rather on a suitable Poisson algebra of gauge invariant observables.[1] For generally covariant theories without matter fields, such observables are necessarily non-local. With matter, local observables may be constructed by using combinations of matter and geometric variables such that the matter acts as a reference system to locate spacetime points.[2] In either case, the emphasis is on first finding gauge invariant classical observables, and attempting to represent these as operators in a quantum theory.*

Dualities are potentially very useful if they allow the probing of one quantum theory using another which is better understood. In this sense dualities may be viewed as a means for obtaining new "collective variables" for studying a given theory in spacetime or energy domains where other variables reach their limits of usefulness. Of particular interest is the question of whether dualties can provide insights for quantum gravity. The AdS/CFT correspondence[4,5] in principle has the potential to do so. However to date it has yielded little insight into what happens to spacetime at the quantum level, or the role played by diffeomorphism invariance in the dual CFT. For this reason it may be useful to search for duals of geometric theories simpler than general relativity or supergravity. An interesting probe of this correspondence would be to seek manifestations in the CFT of interesting metric structures such as event horizons. A more general question in this regard is understanding of the role played in the CFT by metric parameters in asymptotically AdS spacetimes.

In this paper I probe some of these questions. I first describe a rather explicit duality between the simple harmonic oscillator and a topological field theory in four dimensions. Following this I discuss the AdS/CFT duality between gravity on 5-dimensional AdS spacetime and 4-dimensional conformal Yang-Mills theory, in particular the Wilson loop calculation. This calculation may be done explicitly for asymptotically AdS scalar field spacetimes with naked singularities. The results are remarkably similar to those for the AdS black hole. I discuss the interpretation of this result with emphasis on whether the calculation captures true horizon and temperature information in the CFT.

## 2. A Toy Duality

Consider the 4-dimensional topological field theory given by the action

$$S = \int_M B \wedge F(A). \tag{1}$$

The fields are the Abelian connection $A$ and the two-form $B$, with $F(A) = dA$. This action is invariant under diffeomorphisms, $U(1)$ gauge transformations, and $B \rightarrow B + d\Lambda$ for one forms $\Lambda$ (for vanishing surface terms). Consider the Hamiltonian

*An alternative is to construct a quantum theory by first eliminating all gauge invariances at the classical level; the remaining problem here is whether different gauges lead to unitarily equivalent quantum theories.

quantization of this action for manifolds $M \sim \Sigma \times R$ where $\Sigma$ is compact without boundary. [a]Since the action is first order, it is easy to put into canonical from:

$$S = \int_{\Sigma \times R} \epsilon^{0abc} \left[ B_{ab} \partial_0 A_c + 2 B_{0a} F_{bc} - B_{ab} \partial_c A_0 \right]. \tag{2}$$

Thus the canonical coordinates are $(A_a, E^a)$, where $E^a = \epsilon^{0abc} B_{bc}$. The Hamiltonian is a linear combination of the constraints

$$F_{ab} = 0, \qquad \partial_a E^a = 0. \tag{3}$$

These constraints are first class and generate the gauge transformations of the theory. Note that the Hamiltonian is a linear combination of the two contraints as expected for a generally covariant theory, and that spatial diffeomorphisms are generated by the combination $A_a \partial_b E^b + E^b F_{ab}$.

Since the constraints generate gauge transformations, gauge invariant observables $\mathcal{O}(E, A)$ are defined by the Poisson bracket conditions

$$\{\mathcal{O}(E, A), C(E, A)\} = 0, \tag{4}$$

where $C$ denotes the two constraints. In the present case the basic observables satisfying this condition are

$$T^0(A, \gamma) = \exp\left[ \int_\gamma ds \dot{\gamma}^a A_a \right], \qquad T^1(E, S) = \int_S d^2\sigma \; n_a E^a, \tag{5}$$

which are parametrized by loops $\gamma$ and surfaces $S$, $n_a$ is a one form field defining the surface $S$ ($\epsilon^{0abc} n_c$ is the area 2-form and $\dot{\gamma}^a$ is tangent vector to the loop $\gamma$). These observables satisfy the Poisson algebra

$$\left\{ T^0(A, \gamma), T^0(A, \beta) \right\} = 0, \qquad \left\{ T^1(E, S), T^1(E, S') \right\} = 0, \tag{6}$$

$$\left\{ T^0(A, \gamma), T^1(E, S) \right\} = c(\gamma, S) T^0(A, \gamma) \tag{7}$$

where

$$c(\gamma, S) = \int ds \int d^2\sigma \; \dot{\gamma}^a n_a \delta^3(\gamma(s) - S(\sigma)). \tag{8}$$

The last Poisson bracket vanishes if the loop lies in the surface.

On the constraint surface the observables $T^0$ and $T^1$ depend only on the non-contractible loops and surfaces in $\Sigma$. Thus they capture topological information about $\Sigma$. To proceed further we must fix the topology, which determines the number of independent observables, and hence degrees of freedom. For the case $\Sigma \sim S^1 \times S^2$ there is exactly one non-contractible loop and surface, for which $c(\gamma, S) = 1$. Thus there are two degrees of freedom, and the Poisson algebra (7) becomes

$$\left\{ T^0, T^1 \right\} = T^0. \tag{9}$$

[a]Quantization of models of this type have been discussed before. The Hamiltonian approach followed here is a review of earlier work by the author.[3]

A quantum theory can be constructed by representing this algebra on a state space of "occupation numbers" by defining

$$\hat{T}^0|n>=\sqrt{n}|n-1> \qquad\qquad \hat{T}^1|n>=n|n>\,. \tag{10}$$

The commutator algebra following from these defintions has the appropriate classical limit. This establishes the duality with the harmonic oscillator, with $T^1$ corresponding to the "composite" number operator $a^\dagger a$.

This procedure may be applied in other dimensions and spacetime topologies. The number of degrees of freedom depend on the topology. There is an extension of the observables to the non-Abelian case, which is a bit more involved, and the algebra of observables does not correspond to the harmonic oscillator.[3]

The basic lesson from this example is that dualities may turn out to be of no more than mathematical interest. The lesson for quantum gravity is that even if this type of Hamiltonian procedure can be carried out to completion starting from classical general relativity (without matter), the task of extracting a classical spacetime would be formidable. This is because fully gauge invariant observables, being non-local, would not carry any local spacetime information. This problem may be eased by solving the Hamiltonian constraint at the classical level via gauge fixing to a preferred classical time coordinate, or alternatively, by incorporating matter and using it to locate spacetime points in a diffemorphism invariant manner.[2]

## 3. The Wilson loop in AdS/CFT

Dualities between theories with infinitely many degrees of freedom are clearly of much more interest than quantum mechanical models of the type discussed in the previous section. An early example is the Thirring model and its bosonic dual.

A potential basis for constructing dual theories is the basic observation that quantum field theories on a fixed background spacetime carry representations of the spacetime isometry group. Since the possibility exists that spacetimes of different dimensions may have the same spacetime isometry group, theories in different dimensions have the potential to be dual. Since the conformal group of $d$-dimensional Minkowski spacetime and the isometry group of $(d+1)$-dimensional anti-deSitter spacetime are both $SO(d,2)$, there is the potential of a large number of dualities between conformal theories on Minkowski spacetime and theories on AdS, or asymptotically AdS (AAdS) spacetimes. This of course is a first requirement, and it is necessary to establish a correspondence between operators, and between states, in the two theories. For example collective states and composite operators of one theory may correspond to more "basic" states and operators in the other. If the duality is to aid in performing computations in one theory which are difficult in the other, it is necessary to establish the latter correspondences.

The first example of an AdS/CFT duality was discovered by Maldacena.[4] It is between supergravity on AAdS spacetimes and supersymmetric conformal Yang-Mills theory. This has been extensively studied in the last few years[5]. It has the potential to yield insights into quantum gravity, since in principle the dual YM theory provides a window. In practice many key questions remain unanswered, among them the role that spacetime diffeomorphisms play in the YM theory.

One of the results of these works, of interest for this paper, is the proposal for determining the expectation value of the Wilson loop in the Yang-Mills theory via a computation in supergravity.[6] This proposal has been studied in detail by a number of authors, [7,8] and includes discussion of the exciting possibility of correspondences between (quantum) phases of Yang-Mills theory and classical AAdS geometries.

The proposal is the following: consider string worldsheets $s_\gamma$ in an AAdS spacetime which have the loop $\gamma$ as boundary. The expectation value of the Wilson loop is given by

$$< W_\gamma >= \int Ds_\gamma \; e^{-S_{NG}(s_\gamma)} \tag{11}$$

where $S_{NG} = \int d\sigma d\tau \sqrt{h}$ is the Nambu-Goto action of the world sheet.[6,7] (This is reminiscent of the "no-boundary" proposal for quantum gravity.) In practice, the integral is approximated by computing the Nambu-Goto action for a solution of the classical string equations, for a choosen class of surface $s$, bounding loop $\gamma$, and AAdS spacetime.

For static rectangular loops, of space and time extensions $L$ and $T$ lying along the $t, x$ coordinates of the Minkowski boundary, $< W_\gamma >$ determines the quark potential $V(L)$. The calculation in fact gives a divergent $S_{NG}(s_\gamma)$. This is renormalised by subtracting the divergent part of the integral, which is proportional to the loop's circumference. Thus

$$< W_\gamma >= e^{-S_{NG}(L)-kL} \sim e^{-TV(L)}, \tag{12}$$

where $k$ is a constant.

In principle this computation should be capable of yielding any of the three phases of YM theory: $V(L) \sim -1/L$ (Coulomb), $V(L) \sim L$ (confinement) and $V(L) \sim -e^{-L}/L$ (Higgs) depending on the geometry used. In this way the proposal provides a link between AAdS geometries and phases of Yang-Mills theory. In particular, it may also provide a means of seeing how horizon and singularity information is encoded, or at least interpreted in the conformal YM theory.

The calculation was first done for the global AdS geometry,[6] and subsequently for the AdS-Schwarzschild metric.[9] In the former case, one obtains the result expected on grounds of unbroken conformal invariance – the only scale is the quark separation $L$, which results in the Coulomb phase. In the latter case, the black hole horizon provides a scale. The calculation for this geometry shows a distortion of the Coulomb behaviour such that the the potential $V(L)$ goes to zero at a finite value of $L$. This is physically interpreted to be the result of temperature, which screens the $-1/L$ behaviour of the potential for $L$ larger than a critical "screening length" $L_c$. The derivation of this result follows.

The specific form of the AdS black hole metric used is

$$ds^2 = \left(\frac{r}{l}\right)^2 \left[-f(r)dt^2 + (dx^i)^2\right] + \left(\frac{l}{r}\right)^2 f^{-1}(r)dr^2 \tag{13}$$

where $f(r) = 1 - r_0^4/r^4$, and $r_0$ is the black hole horizon parameter.[b] The conformal Minkowski boundary coordinates where the loop lies are $t, x^i$ ($i = 1..3$). A static rectangular loop may be taken to lie along one of the spatial coordinates $x$, and $t$. The action is calculated by setting $\tau = t$ and $\sigma = x$, such that $r = r(x)$ describes the embedding of the worldsheet in the black hole geometry. The Euclidean metric is used, which gives

$$S_{NG} = T \int dx \sqrt{(\partial_x r)^2 + (r^4 - r_0^4)/l^4} \tag{14}$$

This is extremised by noting that it doesn't depend explicitly on $x$, so the "energy"

$$e = \frac{r^4 - r_0^4}{\sqrt{(\partial_x r)^2 + (r^4 - r_0^4)/l^4}} \tag{15}$$

is conserved. It is useful to write the constant $e$ as a function of the minimum value $r_m$ of $r(x)$. Thus $r_m$ is given by $e = l^2\sqrt{r_m^4 - r_0^4}$. Integrating the last equation gives an integral for $x(r)$:

$$x(r) = \frac{l^2}{r_m}\sqrt{1 - \left(\frac{r_0}{r_m}\right)^4} \int_1^{r/r_m} \frac{dy}{\sqrt{(y^4-1)(y^4-(r_0/r_m)^4)}}. \tag{16}$$

As $r \to \infty$ (ie. to the boundary), this integral gives half the spatial length $L$ of the loop (since it is symmetric in $r$). Thus

$$L(r_m) = \frac{2l^2}{r_m}\sqrt{1 - \left(\frac{r_0}{r_m}\right)^4} \int_1^{\infty} \frac{dy}{\sqrt{(y^4-1)(y^4-(r_0/r_m)^4)}}, \tag{17}$$

which relates the loop dimension $L$ to the minimum value $r_m$ of $r$. The action for this solution is

$$S_{NG}|_{\text{loop}} = T r_m \int_1^{\infty} dy \sqrt{\frac{y^4 - (r_0/r_m)^4}{y^4 - 1}}. \tag{18}$$

This integral is divergent and proportional to $y$ for large $y$. It is rendered finite in the usual way by integrating up to $y_{max}$, subtracting the divergence $r_m y_{max}$ (up to an additive constant to be determined), and then removing the regulator $y_{max} \to \infty$. There is a physical interpretation of this procedure which provides the additive constant: the subtracted term corresponds to the energy of free quarks, which is

[b] The $S^5$ part of the 10−dimensional metric and the string scale $\alpha'$ also appear in the metric but are not relevant for the main outline of the calculation. The AdS scale $l$ is related to the YM parameters $(g, N)$ by $l^2 = \sqrt{4\pi g N}$.

also divergent, and given by the static configuration where $r$ is not a function of $x$. This solution is obtained by setting $\tau = t$ and $\sigma = r$ (with all other coordinates constant) in the Nambu-Goto action: $S_{NG}|_{\rm free} = \int_{r_0}^{\infty} dr$, where the lower limit is the horizon radius, since this is the lowest value of $r$ in the Euclidean calculation. The finite potential $V(r_m)$ is therefore

$$V(r_m) = r_0 - r_m + r_m \int_1^{\infty} dy \left( \sqrt{\frac{y^4 - (r_0/r_m)^4}{y^4 - 1}} - 1 \right) \tag{19}$$

The function $V(L)$ is obtained by substituting $L(r_m)$ from Eq. (17) into the last equation, which may be done numerically. Figure 1. shows the typical behaviour: the screening length $L_c \sim 0.76$ corresponds to the intersection of the graph with the $L$ axis. The small $L$ behavior is $\sim -1/L$, and the result is considered unphysical above the $L$ axis. For global AdS ($r_0 = 0$), $V(L) \sim -1/L$ for all $L$.

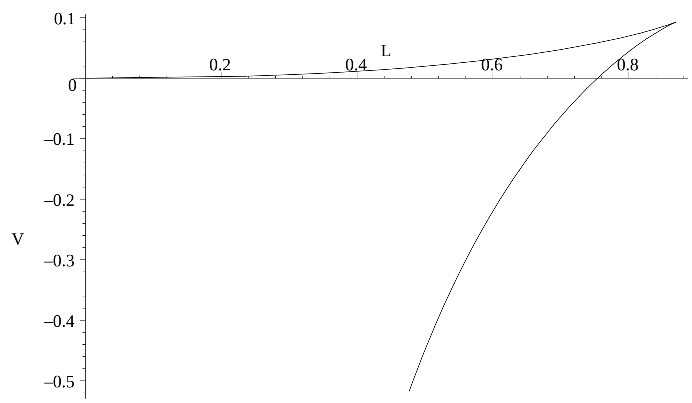


Fig. 1. $V(L)$ for the positive mass AdS black hole for $r_0 = l = 1.0$.

## 4. Naked singularities

From the AdS black hole result, it appears that dimensional parameters in metrics will lead to distortions of the Coulomb behaviour of $V(L)$. The natural question is whether spacetimes other than black holes can lead to qualitatively similar behaviour of $V(L)$, and if so, what the YM interpretation should be. This is the question we now probe by using AAdS spacetimes that have naked singularities.

We first give a new solution of the Einstein-scalar field equations for massless minimally coupled scalar field with negative cosmological constant. This solution is used as the AAdS geometry for the Wilson loop calculation, and is compared with the result for the negative mass AdS-Schwarzschild metric.

The low energy actions derived from string theory are of the form

$$S = \int d^d x \, \sqrt{-g} \left[ e^{-\phi} \left( R + (\nabla\phi)^2 \right) + \Lambda + L_{\text{matter}} \right]. \tag{20}$$

Such actions may be put into the standard minimal coupling form by the conformal transformation of the metric $g \to g\exp(-\phi)$. Thus we can consider the usual Einstein-scalar equation in $d$ dimensions

$$G_{ab} - \frac{(d-1)(d-2)}{2l^2} g_{ab} = \partial_a\phi\partial_b\phi - \frac{1}{2} g_{ab}\partial_c\phi\partial^c\phi. \tag{21}$$

The new solution is obtained starting with the metric ansatz

$$ds^2 = \left(\frac{r}{l}\right)^2 \left( -f(r)dv^2 + \sum_{i=1}^{d-2} (dx^i)^2 \right) + 2g(r)dvdr \tag{22}$$

where $r, v$ are radial and advanced time coordinates. The solution is

$$f(r) = 1, \qquad g(r) = \left[ 1 + \left(\frac{r_0}{r}\right)^{2(d-1)} \right]^{-1/2} \tag{23}$$

with scalar field given by

$$\partial_r\phi(r) \sim \frac{1}{r} \left[ 1 + \left(\frac{r}{r_0}\right)^{2(d-1)} \right]^{-1/2}, \tag{24}$$

where the proportionality factor depends on $d$. For $r_0 = 0$ the solution is global AdS in these coordinates – the scalar field vanishes with $r_0$. In all dimensions $d \geq 3$, the large $r$ behavior indicates that the spacetimes are asymptotically AdS, with fairly rapid falloff with $r$. It is also evident that there is a timelike curvature singularity at $r = 0$. The conformal Minkowski boundary is obtained by multiplying the solution by the factor $(l/r)^2$ and taking the limit $r \to \infty$.

We consider the case $d = 5$, and describe the Wilson loop calculation using world sheets in this geometry. The key similarities and differences from the black hole case will become evident. As for the AdS black hole, consider the surface given by $r = r(x)$ with $\sigma = x$ and $\tau = v$. The action is

$$S_{NG} = T \int dx \sqrt{ \left(\frac{r}{l}\right)^4 + (\partial_x r)^2 \left[ 1 + \left(\frac{r_0}{r}\right)^8 \right]^{-1} }. \tag{25}$$

On the solution this is

$$S_{NG}|_{\text{loop}} = T r_m \int_1^\infty \frac{y^6 dy}{\sqrt{(y^4-1)(y^8 + (r_0/r_m)^8)}}, \tag{26}$$

where $r_m$ is the minimum value of $r(x)$ for this worldsheet. The dimension $L$ of the rectangular loop is related to $r_m$ by

$$L(r_m) = \frac{2l^2}{r_m} \int_1^\infty \frac{y^2 dy}{\sqrt{(y^4-1)(y^8 + (r_0/r_m)^8)}}. \tag{27}$$

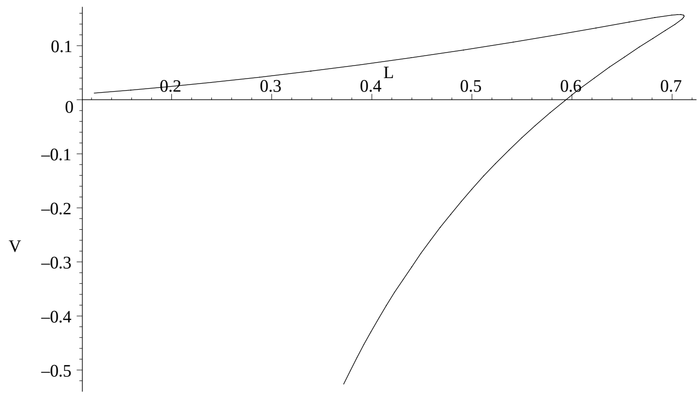


Fig. 2. $V(L)$ for the singular scalar field solution for $r_0 = l = 1.0$.

The action and hence $V(r_m)$ are divergent on the solution, with the divergence again proportional to $y$ for large $y$. Following the same renormalization procedure, we subtract the divergent action corresponding to free quarks. This is the configuration $\tau = v$ and $\sigma = r$ ($\neq r(x)$) with all the $x^i$ constant. Its action is

$$S_{NG}|_{\text{free}} = T \int_0^\infty dr\ g_{vr} = \int_0^\infty dr \left[1 + \left(\frac{r_0}{r}\right)^8\right]^{-1/2}. \qquad (28)$$

This differs from the constant integrand obtained for the AdS black hole. The finite potential is obtained as before by integrating the loop and free actions up to a finite $y_{max}$, subtracting, and taking the limit $y_{max} \to \infty$. This gives

$$V(r_m) = r_m \int_1^\infty \frac{y^4 dy}{\sqrt{y^8 + (r_0/r_m)^8}} \left[\frac{y^2}{\sqrt{y^4 - 1}} - 1\right] - r_m \int_0^1 dy \frac{y^4}{\sqrt{y^8 + (r_0/r_m)^8}}. \qquad (29)$$

The last equation combined with Eq. (27) gives $V(L)$ for the scalar field solution. A typical plot of $V(L)$ is shown in Fig. 2. The similarity with the AdS black hole case is apparent, in particular the intersection with the $L$ axis at a critical $L_c$. Also similar is that $V(L)$ becomes double valued above this axis, which again indicates that the calculation is not valid for $L > L_c$.

The comparison of these results with the negative mass AdS-Schwarzschild case is interesting.[c] The calculation proceeds in exactly the same way with the only difference being the subtraction integral necessary to obtain a finite $V(L)$: this now has lower limit $r = 0$ rather than the horizon radius $r = r_0$. The result is

$$V(r_m) \quad = \quad r_m \left[-1 + \int_1^\infty dy \left(\sqrt{\frac{y^4 + (r_0/r_m)^4}{y^4 - 1}} - 1\right)\right], \qquad (30)$$

[c] I thank T. Padmanabhan for asking about this case.

$$L(r_m) \quad = \quad \frac{2l^2}{r_m}\sqrt{1+\left(\frac{r_0}{r_m}\right)^4}\int_1^\infty \frac{dy}{\sqrt{(y^4-1)(y^4+(r_0/r_m)^4)}}. \qquad (31)$$

The graph of $V(L)$ appears in Fig. 3. The features similar to the previous cases are again the $-1/L$ behaviour for sufficiently small $L$, and the intersection with the $L$ axis. The additional feature is that the potential does not become double valued for $L > L_c$, and thus may represent real physics. Remarkable is the linear behavior in this region, which indicates a confining potential. Thus this singular spacetime seems to correspond to the Coulomb to confining phase transition!

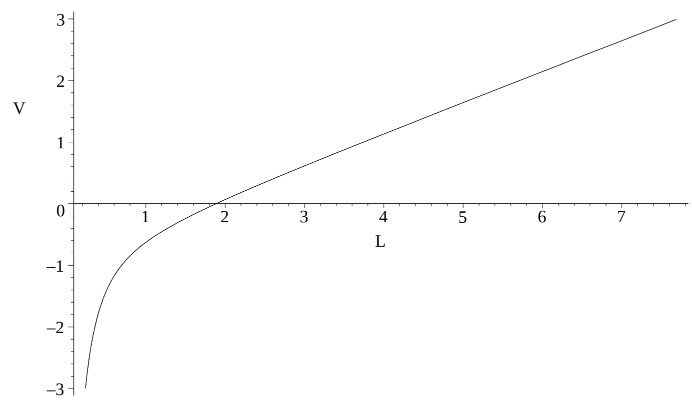


Fig. 3. $V(L)$ for the negative mass AdS black hole for $r_0 = l = 1.0$.

## 5. Conclusions

We have seen that there are explicit dualities between a class of topological field theories and the simple harmonic oscillator. These appear to be mainly of mathematical interest since the correspondences are in a sense too limited to extract "physics." The model nevertheless raises the question of what physics can be learned about geometrical theories from their non-geometrical quantum duals.

The purpose of the AdS/CFT Wilson loop calculation for singular spacetimes is to see how their metric parameters are manifested in the YM quark potential. A component of this question is whether the calculation even leads to sensible results.

The surprising answer is that the distortion of the Coulomb behaviour for the AdS black hole is also present in the singular scalar field and negative mass AdS-Schwarzschild spacetimes, and it is qualitatively very similar: a "thermal screening" interpretation is clearly possible by looking at just the potential for $L \leq L_c$.

The negative mass AdS-Schwarzschild metric presents an additional surprise: it suggests a Coulomb to confining phase transition at $L = L_c$. Why a nice result like this follows from such a bad spacetime requires further probing of the AdS/CFT conjecture.

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